Noisy all-pay auctions

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Abstract

We study all-pay auctions with noise both theoretically and experimentally. First we show theoretically that all-pay auctions with noise can be surprisingly similar to standard all-pay auctions with complete information in several important respects. Specifically, equilibria in both formats feature the same *expected* expenditures. Without noise, predictions are in mixed strategies, while with noise, predictions are in pure strategies. We then report the results of an experiment on our model and find qualitative support for the predicted equivalence of *expected* expenditures. Our experimental results on the noisy version of the model are particularly interesting, however, as subjects' behavior is more closely in line with the model's theoretical equilibrium predictions than the treatment without noise in terms of reduced variance.

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1 Introduction

Though competitors can usually choose their own effort levels with certainty, luck is often a central element in determining the finished product or final output that is judged in contests. For example, when researchers try to find a new treatment for a specific disease, they often expend considerable resources studying a candidate drug which is subsequently revealed to have no therapeutic effect. Likewise, effective treatments are sometimes stumbled upon by researchers who expended surprisingly little effort.[1] However, the burgeoning contest literature largely ignores the effect of luck, and assumes that contestant effort maps deterministically into contestant output.

One prominent example of such a model is the all-pay auction. In such auctions, contestants simultaneously choose expenditures in an attempt to win a prize; all spending is forfeited regardless of who wins and the contestant with the highest expenditure wins the prize with certainty.[2] Thus, there is no role for luck in mapping expenditures into output. Nor is there any role for luck in mapping outputs into prize allocation.

In this paper, we consider all-pay auctions in which luck pays a prominent role in determining the output of a contestant. In particular, each contestant's output is her chosen expenditure multiplied by the realization of a noise parameter that is not observed until after expenditures are chosen.[3]

We focus on multiplicative noise for several reasons.[4] First, with additive noise, a contestant who chooses an expenditure of zero can end up with positive output. In our view a model in which a contestant with a positive expenditure can lose to one with an expenditure of zero is undesirable. Second, additive noise can result in negative outputs, which are hard to interpret.[5]

When solving for equilibrium, different distributions of noise lead to dramatically different predictions. Since we are interested in incorporating uncertainty into the link between expenditures and output, we opt

[1]Perhaps the most famous example is the discovery of penicillin.

[2]In the event of a tie the prize is usually assumed to be assigned randomly to one of the highest-bidding contestants.

[3]Note that by focusing on the all-pay auction, we are able to isolate the effect of luck linking expenditures and outputs. Luck that focuses on prize allocation for given outputs is removed from the model. This is in contrast to the literature on Tullock contests.

[4]With multiplicative noise, you can take any monotonic transformation of the output function, and the game is strategically equivalent. This includes taking the natural log of output, which would allow you to additively separate expenditures and noise. However, this would lead to undefined output at an expenditure of zero.

[5]See, also, Jia (2008) for a discussion of the merits of multiplicative noise versus additive noise.

to focus on the case with maximum entropy, in which the noise is uniformly distributed. This interesting case is also the most straightforward to take to the laboratory.

We derive equilibrium and find striking similarities between equilibrium predictions in all-pay auctions with and without noise, in expectation. In our all-pay auction with noise and $n \geq 2$ bidders, there always exist equilibria in which only the strongest two contestants choose positive expenditures. The strongest contestant receives an expected payoff equal to the difference between his valuation and that of the second-strongest contestant, and the second-strongest receives an expected payoff of zero, as do all of the inactive contestants. If contestants have a common valuation for the prize, rents are fully dissipated. All of these features, as well as the total expected level of spending by the two active bidders, are mirrored in equilibria of the standard all-pay auction without noise. The only difference is that equilibrium is in pure strategies in the model with noise, and in mixed strategies in the model without noise.

We experimentally test these predictions for the simple case of two symmetric contestants. Our three treatments are: 1) an all-pay auction without noise; 2) an all-pay auction with multiplicative noise uniformly distributed on $[0, 1]$; 3) an all-pay auction with multiplicative noise uniformly distributed on $[0, 2]$. The last of these treatments is designed to determine whether or not luck that can increase output relative to expenditure differs (luck can be good or bad) from an environment where luck can only decrease effort relative to expenditures. Our results are qualitatively in line with predictions. In particular, average expenditures do not differ between the three treatments.

An interesting difference in the results of the treatments, however, is that the variance in observed expenditures is significantly reduced by the introduction of noise. Indeed, the large mass point of zero expenditures that is typically observed in all-pay auctions without noise is dramatically reduced when noise is added. Though we can not offer any definite explanation for this, the addition of noise does seem to lead subjects to focus more toward equilibrium bidding behavior

We are not the first to consider the impact of stochastic elements on an all-pay auction. In one of the most famous treatments of internal labor contests, for example, Lazear and Rosen (1981) model employees competing for compensation rewards as an all-pay auction in which efforts are influenced by an additive stochastic term. More recently, Konrad (2009) provides a general analysis of the standard (simultaneous) all-pay auction with additive noise, and Segev and Sela (2014) consider sequential all-pay auctions with additive noise.

In outlining the similarities and differences between the all-pay auction with and without noise, we contribute to an important strand of literature that compares contest formats. One paper related to ours, for example, is Jia (2008), who also considers an all-pay aucti1on with multiplicative noise. The key distinction between the two papers, however, is that Jia (2008) assumes that noise is distributed according to what he terms as an inverse exponential distribution, a generalized form of the Frechét distribution. With that assumption, the all-pay auction with noise becomes strategically equivalent to a contest with the well-known ratio-form contest success function (CSF), introduced by Tullock (1980) and axiomatized by Skaperdas (1996). In our paper the primary assumption is that noise is distributed uniformly; however, we also extend Jia's result to show that such contests also contain equilibria that are equivalent to those in standard all-pay auctions.[6]

Other papers establishing relationships between various contest formats include Che and Gale (2000), who show that an all-pay auction with additive noise can be thought of as a contest with a difference-form contest success function, a format studied by Hirshleifer (1989) and Baik (1998). Baye and Hoppe (2003), meanwhile, establish equivalence results between stochastic innovation games and Tullock-style contests. Alcalde and Dahm (2007) develop a new contest format they refer to as the serial contest. They show that equilibria of the serial contest have many similar properties in common with the standard all-pay auction. Baye, Kovenock and De Vries (2012) provide a general framework to relate a large family of two-player rank-order contests that allow for various types of externalities, such as all-pay auctions with feelings of regret, and Chowdhury and Sheremeta (2014) provide a general framework to relate Tullock-style contest formats that appear different but are in fact strategically equivalent.

[6] Since this is not our primary focus, these results have been relegated to Appendix B.

2 Theoretical Predictions

A group of $m \geq 2$ contestants simultaneously choose non-negative expenditures in an attempt to win a prize. The value of the prize to contestant i is $V_i > 0$, and this is common knowledge. The expenditure of contestant i is denoted by x_i. All expenditures are paid regardless of who wins the prize. In allocating the prize, the auction designer cannot directly observe contestant expenditures. Rather, they observe a final output from each contestant that is the product of the contestant's expenditure and a random factor. The output level of contestant i is

$$q_i = \theta_i x_i.$$

The prize is allocated to the contestant with the highest output level.[7]

The parameter θ_i represents contestant i's random luck factor. These factors are assumed to be independently and uniformly distributed on $[0, 1]$, and contestants know only those distributional characteristics.[8] Contestants do not know the value of their own luck factor for certain, nor the parameters of their competitors, when making their expenditure choice. However, they do know the distribution from which these parameters are drawn. Contestant i's objective function is then

$$\max_{x_i \geq 0} \mathbb{E}U_i = \text{Pr}(q_i > \max_{j \neq i}\{q_j\})V_i - x_i,$$

where $\text{Pr}(q_i > \max_{j \neq i}\{q_j\})$ represents the probability that contestant i's output level is the highest—that is, the probability that contestant i wins the prize.

Note that monotonic transformations of the production technology do not technically alter the strategic aspects of the game, though this statement comes with important caveats. First, although a logarithmic transformation of our model would yield an all-pay auction with additive noise, such a transformation would require a restricted support (excluding expenditures of zero). Further, the choice of which distribution the noise parameter follows is crucial to analyzing any version of the model, as illustrated by Fu

[7] The possibility of ties will be ruled out by the distributional assumptions on θ_i as long as some agents choose positive expenditure levels. In the event that all agents choose to spend zero, we follow the norm in the literature and assume the prize is allocated by a fair lottery.

[8] For simplicity, we assume the upper bound of the support of this distribution is 1. All our results hold without this simplification.

and Lu (2010), who also study noisy all-pay contest models. Any characterizations of equilibria will be directly dependent on the distributional assumptions of the noise involved, as also noted by Konrad (2009) for the case of standard additive noise models. Taking the log of a uniform random variable does not result in contestants facing the same uniform noise, which alters the model conceptually both in terms of theoretical derivations and in terms of how lab subjects view/interpret the strategic environment they face. To further illustrate this importance, we also find equilibrium in all-pay auctions in which the noise follows an extreme value distribution. See Appendix B. We show that such a noisy all-pay auctions are equivalent to contests with a ratio-form contest success function.

To determine each contestant's probability of winning, we let n denote the number of active bidders—that is, the number of contestants who choose to expend a strictly positive amount. We then note that for any given contestant i, without loss of generality we can relabel the active contestants such that $i = n$, and the remaining contestants are ordered such that $x_1 \leq x_2 \leq \ldots \leq x_{n-1}$. Based on the new labels, x_n is then the expenditure of the contestant whose probability of victory we are interested in, and we then define parameters $c_j = \frac{x_j}{x_n}$ for $1 \leq j \leq n-1$. We then have that $0 < c_1 \leq c_2 \leq \ldots \leq c_{n-1}$, and we denote by k the threshold index at which $c_k \leq 1$ and $c_{k+1} > 1$. Possible values for k are $0, 1, 2, \ldots, n-1$, and the typical case is $c_1 \leq c_2 \leq \ldots \leq c_k \leq 1 \leq c_{k+1} \leq \ldots \leq c_{n-1}$. After this relabeling, the probability of winning, conditional on the vector of expenditure levels can be explicitly calculated.

Theorem 1. With agents relabeled as specified above, $\Pr(q_n > \max_{j \neq n}\{q_j\}) =$

$$\begin{cases} \prod_{j=1}^{n-1} \frac{1}{c_j} \frac{1}{n} & \text{if } k = 0 \\ \prod_{j=1}^{n-1} \frac{1}{c_j} \frac{c_1^n}{n} + \prod_{j=2}^{n-1} \frac{1}{c_j} \frac{c_2^{n-1} - c_1^{n-1}}{(n-1)} + \ldots + \prod_{j=k}^{n-1} \frac{1}{c_j} \frac{c_k^{n-k+1} - c_{k-1}^{n-k+1}}{(n-k+1)} + \prod_{j=k+1}^{n-1} \frac{1}{c_j} \frac{1 - c_k^{n-k}}{(n-k)} & \text{if } 1 \leq k \leq n-2 \\ \prod_{j=1}^{n-1} \frac{1}{c_j} \frac{c_1^n}{n} + \prod_{j=2}^{n-1} \frac{1}{c_j} \frac{c_2^{n-1} - c_1^{n-1}}{(n-1)} + \ldots + \frac{1}{c_{n-1}} \frac{c_{n-1}^2 - c_{n-2}^2}{2} + (1 - c_{n-1}) & \text{if } k = n-1 \end{cases} \tag{1}$$

With these probabilities in hand, we turn attention to determining equilibrium expenditures. Interestingly, there will always exist an equilibrium in which the two contestants with the highest values chose positive expenditures, while the others choose expenditures of zero.

Theorem 2. For an all-pay auction with noise where $V_1 \geq V_2 \geq \ldots \geq V_n$, there exists a Nash equilibrium in which $x_1^* = \frac{V_2}{2}$, $x_2^* = \frac{V_2^2}{2V_1}$, and $x_j^* = 0$ for all $j > 2$; expected payoffs are $\mathbb{E}U_1^* = V_1 - V_2$, $\mathbb{E}U_2^* = 0$, and $\mathbb{E}U_j^* = 0$ for all $j > 2$; and total expenditures are $\sum_{j=1}^{n} x_j^* = \frac{V_2(V_1+V_2)}{2V_1}$.

Proof: See Appendix A.

This equilibrium is strikingly similar to equilibrium in all-pay auctions without noise[9] In fact, without noise, there exists an equilibrium where behavior is equivalent in expectation. The only difference is that for the two active contestants equilibrium expenditures are not in pure strategies.

In all-pay auctions with and without noise, there can be a multiplicity of equilibrium when $n > 2$. Our primary focus is in testing the prediction that, in expectation, expenditures are equivalent between all-pay auctions with and without noise. As such, we turn attention to the $n = 2$ case in which the equilibrium described in Theorem 2 is unique.

Proposition 1. In a two-player all-pay auction with noise with $V_1 \geq V_2$ there exists a unique pure-strategy Nash equilibrium. In that equilibrium the contestants' expenditure levels are $x_1^* = \frac{V_2}{2}$ and $x_2^* = \frac{V_2^2}{2V_1}$, their expected payoffs are $\mathbb{E}U_1^* = V_1 - V_2$ and $\mathbb{E}U_2^* = 0$, and total expenditures are $(x_1^* + x_2^*) = \frac{V_2(V_1+V_2)}{2V_1}$.

Proof: See Appendix A.

These equilibrium features match those of the expected bids, payoffs, and total expenditures in any equilibrium of the two-player all-pay auction with complete information as analyzed by Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996). Thus, for the two-player case, an all-pay auction with noise has the same expected expenditure levels, payoffs, and total spending as the all-pay auction without noise. In the case that $V_1 = V_2$ we also get the corollary result that rents are fully dissipated as in the standard all-pay auction.

3 Experimental design

In each experimental session participants are anonymously and randomly matched into pairs. This matching is fixed for all forty periods of the experiment.[10] In every period each pair participates in a possibly

[9]See Baye, Kovenock, and de Vries (1996) for a characterization of equilibrium in all-pay auctions without noise.

[10]When participants are in an all-pay auction without noise, and a random re-matching protocol is employed, a participant could submit the same expenditure in successive rounds without the opponent learning to best respond by submitting a

noisy all-pay auction. The value of the prize is $USD5$ for both contestants, and this is common knowledge. Contestants choose unrecoverable expenditures in private. Each contestant's expenditure is multiplied by a parameter, and this product is referred to as their output. The contestant with the highest output wins the prize, with ties resolved by a fair randomization.

The parameter is determined in three different ways, and these define the three treatments in our between-subject experimental design. In the baseline treatment the parameter is equal to one for all contestants, so that the game is a standard all-pay auction. In the other two treatments each contestants' parameter is an iid draw from a uniform distribution. In the $[0, 1]$ treatment, the support of the uniform distribution is $[0, 1]$, while in the $[0, 2]$ treatment the support is $[0, 2]$.

The upper bound of the support is theoretically irrelevant. However, in the $[0, 1]$ treatment, a contestant's output is almost surely less than their expenditure, and contestants may perceive this as only being exposed to bad luck. The $[0, 2]$ treatment allows us to determine whether having noise that can increase output relative to expenditures is behaviorally important.

At the end of a period, contestants observe feedback in the form of their expenditure, parameter, output and payoff. They also observe those of their opponent. Further, the feedback received in each previous period is always on their screens.

In each of our three treatments there are 20 pairs.[11] All sessions were run at the Economic Science Institute at Chapman University. Subjects interacted only via a computer interface, which was programmed in z-Tree (Fischbacher, 2007). At the beginning of an experimental session participants were seated at computers separated by dividers to ensure the privacy of all decisions. All subjects were provided with a hard copy of the instructions, which were read aloud.[12] Afterwards, each participant completed a short quiz to ensure understanding. Once the experiment was complete, participants were paid in private.

Participants earned a $\$7$ show-up fee. Each participant started the experiment with $\$25$ to cover potential losses. Of the 40 periods of the experiment, 10 were randomly selected for payment.[13] Thus,

marginally higher expenditure. Since a key prediction we test is whether or not the introduction of noise results in participants employing pure strategies, the fixed matching protocol is an important feature of our experimental design.

[11] In the baseline and $[0, 2]$ treatments there are two sessions with twenty participants each. In the $[0, 1]$ treatment, there is one session with twenty participants, and two sessions with ten participants each.

[12] The instructions for the $[0, 1]$ treatment can be found in Appendix C.

[13] One participant in the $[0, 1]$ treatment lost more than the starting balance over the course of the ten periods selected for

participants left a session with the sum of their show-up fee ($\$7$), their starting balance ($\$25$) and the sum of their payoffs in each of the 10 periods selected for payment. Payments were rounded up to the nearest $\$0.25$. Participants earned an average of $\$40.41$, with a minimum of $\$7$, and a maximum of $\$70.25$.

4 Results

Since the behavior of participants in a given pair is likely to be correlated, we use pairs as our independent unit of observation in our nonparametric tests. That is, we average the behavior of both contestants in a pair across all periods, and treat this average as an independent observation. This means that we have twenty independent observations per treatment. For each nonparametric test, we report p-values from two tailed tests. This approach is conservative. However, our results are robust to parametric analysis. Specifically, if we treat our data as a balanced panel, and cluster standard errors at the individual pair, our results are unchanged.

4.1 Individual expenditures

Table 1 contains summary statistics of individual expenditures broken down by treatment, as well as the corresponding summary statistics for the Nash equilibrium predictions. The most striking theoretical prediction is that the addition of noise in an all-pay auction will not have any effect on average expenditures. Our results are in line with this prediction: we find no statistically significant difference between average expenditures in all-pay auctions without noise and those with noise, regardless of whether the noise is distributed on $[0, 1]$ (Mann-Whitney, $z = 0.352$, $p = 0.725$), or on $[0, 2]$ (Mann-Whitney, $z = 0.920$, $p = 0.358$).

Furthermore, contrary to our hypothesis, changing the upper bound of the support on the noise does not change average expenditures (Mann-Whitney, $z = -0.568$, $p = 0.570$). That is, contestants do not change their behavior, on average, when the noise is distributed such that their output could exceed their expenditure.

payment. Our analysis is robust to dropping the data from the corresponding pair.

We average expenditures of pairs over all forty periods for out nonparametric tests. Since contestants have the opportunity to learn over the course of the experiment, we now turn attention to how average expenditures change over time. Figure 1 illustrates that the average expenditures are fairly stable over time. Note that average expenditures are below equilibrium predictions, and do not seem to be increasing over the course of the experiment. Indeed, expenditures are significantly below equilibrium predictions in all three treatments.[14]

The fact that behavior does not approach equilibrium, even in the final rounds of the experiment is surprising.[15] However, it is important to note that average under-expenditure is not unusual in two-player complete information all-pay auctions. Such behavior is observed by Potters and De Vries (1998); Gelder et al. (2015); Ernst and Thöni (2013); Kovenock and Rentschler (2020). In addition, our instructions describe expenditures in a novel way. In particular, we use the word "expense" as a way of making the irrevocable nature of expenditures salient to participants. One possibility is that this tends to reduce expenditures. Given the widespread interest in explaining deviations from equilibrium in all-pay auctions (see e.g., the survey Dechenaux et al. (2015) for more on this literature), we view investigating the framing of expenditures as a promising avenue for future research.

While average expenditures are predicted to be equivalent across all three treatments, the predicted distributions of expenditures are dramatically different. When there is not noise, contestants are predicted to randomize uniformly on $[0, 5]$. When there is noise, contestants are predicted to employ a pure strategy of $\$2.50$.

Since the predicted standard deviation of a pure strategy is zero, it is not surprising that the standard deviation of observed expenditures exceeds predictions when there is noise, regardless of whether the noise is on $[0, 1]$ (Sign test, $x = 20$, $p = 0.000$) or on $[0, 2]$ (Sign test, $x = 20$, $p = 0.000$). Interestingly, when there is no noise, we are unable to reject the hypothesis that the standard deviation of observed expenditures conforms to theory (sign test, $x = 11$, $p = 0.824$).

[14] Baseline treatment: Sign test, $x = 16$, $p = 0.012$; $[0, 1]$: Sign test, $x = 17$, $p = 0.003$; $[0, 2]$: Sign test, $x = 16$, $p = 0.012$.

[15] One possible explanation is implicit collusion on the part of contestants. To assess this possibility we plotted both expenditures over the forty periods of the experiment for each pair. We find only three instances in which expenditures are consistent with tacit collusion, and our results are robust to dropping these three observations. The sixty plots just described can be found in the Supplemental Materials.

To illustrate that the observed distribution of expenditures when there is not noise is not uniform, we refer to Figure 2, which contains kernel density plots of expenditures for each treatment. Note that, as is typically observed in the literature, we observe a large mass point at zero in the treatment without noise. We also observe a substantial number of aggressive expenditures, although this is less pronounced.[16]

Figure 2 also demonstrates that there is considerably less dispersion in treatments with noise. In particular, we find that the standard deviation of expenditures is lower when there is no noise then when noise is distributed on $[0, 1]$ (Mann-Whitney, $z = -3.544$, $p = 0.000$) or when noise is distributed on $[0, 2]$ (Mann-Whitney, $z = -3.544$, $p = 0.000$). However, we find no statistically significant difference in the standard deviation of expenditures between the two treatments with noise (Mann-Whitney, $z = 0.730$, $p = 0.462$).

While contestants do not employ the predicted pure strategy equilibrium in auctions with noise, our results are qualitatively in line with theory. The average expenditures do not differ across treatments, and there is considerably less dispersion in treatments with noise. To determine whether these qualitative features persist throughout the experiment, we construct boxplots of expenditures over periods for each treatment. These boxplots are contained in Figure 3. Note that, even in the last periods of the experiment, expenditures at or close to zero are rare in treatments with noise, relative to the case without noise. Further, the increased dispersion we observe without noise also persists throughout the experiment.

5 Discussion

In this paper, we have demonstrated striking similarities in equilibrium predictions between all-pay auctions with multiplicative noise and standard all-pay auctions. Specifically, when noise is distributed uniformly the two models always feature equilibria that are equivalent in the sense of expected spending levels and payoffs.

The multiplicative nature of noise and the distributional assumptions of our model may also provide some intuition into the mechanism behind our equivalence results. In the all-pay auction model without

[16] For a detailed discussion on the observed distribution of expenditures in all-pay auctions see Sheremeta (2013).

noise, pure-strategy equilibria are not possible. Equilibrium strategies are always mixed, and are uniformly distributed over the intervals necessary to keep contestants indifferent. The winner-take-all nature of the standard format requires some degree of randomness to prevent profitable deviations. When noise is present it plays a similar role, allowing for pure-strategy equilibria to exist, but maintaining most of the same characteristics as equilibria in the standard model. In fact, because noise allows pure-strategy equilibria to exist when they otherwise would not, it could perhaps help simplify contestants' decision making processes without altering expected outcomes, and contest designers may prefer to allow for equilibria with uncertain equilibrium spending.

As observed experimental behavior in all-pay auctions is typically far from predictions, it is difficult to know, *a priori* if the predicted similarities will emerge experimentally. However, we find strong evidence that the predictions obtain within our experimental treatment. In particular, expenditures do not significantly differ, on average, between all-pay auctions with noise, and those without noise. Further, the variance of observed expenditures is dramatically reduced by the addition of noise, as is the large number of expenditures at zero.

Whether a multiplicative noise factor is more or less reasonable to consider than an additive factor is debatable, but we believe our results here, along with those of Jia (2008), suggest that such a model is well worth considering. Furthermore, as Jia (2008) notes, multiplicative noise has the advantage of making sure that inactive contestants have no chance of winning, a feature lacking in the case of additive noise. Moreover, contestants can produce negative output under additive noise, which is not possible under multiplicative noise. Negative contest output is difficult to interpret and—for all pay auctions with noise—presents difficulties in CSF interpretation.

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A Proofs

Proof of Theorem 1. We are looking for a closed form expression to $\Pr(\theta_n x_n > \max_{1\leq j\leq n-1}\{\theta_j x_j\})$, which is equal to $\Pr(\theta_n > \max_{1\leq j\leq n-1}\{\theta_j c_j\})$ with $c_j = \frac{x_j}{x_n}$. With the assumptions of uniformity and independence,

$$\Pr(\theta_n > \max_{1\leq j\leq n-1}\{\theta_j c_j\}) = \int_0^1 \Pr(c_1\theta_1 \leq \theta, \ldots, c_{n-1}\theta_{n-1} \leq \theta | \theta_n = \theta)\mathrm{d}\theta = \int_0^1 \prod_{j=1}^{n-1} \Pr(\theta_j \leq \frac{\theta}{c_j})\mathrm{d}\theta.$$

With agents relabeled and k defined as above, and recalling that for any $c > 0$

$$\Pr(\theta_j \leq \frac{\theta}{c}) = \begin{cases} 0 & \text{if } \theta < 0 \\ \frac{\theta}{c} & \text{if } 0 \leq \theta \leq c \\ 1 & \text{if } \theta > c \end{cases}$$

we then have that $\Pr(\theta_n > \max_{1\leq j\leq n-1}\{\theta_j c_j\}) =$

$$\int_0^{c_1} \frac{\theta}{c_1}\frac{\theta}{c_2}\cdots\frac{\theta}{c_{n-1}}\mathrm{d}\theta + \int_{c_1}^{c_2} 1\frac{\theta}{c_2}\cdots\frac{\theta}{c_{n-1}}\mathrm{d}\theta + \ldots + \int_{c_{k-1}}^{c_k} 1\ldots 1\frac{\theta}{c_k}\frac{\theta}{c_{k+1}}\cdots\frac{\theta}{c_{n-1}}\mathrm{d}\theta + \int_{c_k}^{1} 1\ldots 1\frac{\theta}{c_{k+1}}\cdots\frac{\theta}{c_{n-1}}\mathrm{d}\theta \tag{2}$$

for $1 \leq k \leq n-2$. Integrating (2), we obtain $\Pr(\theta_n > \max_{1\leq j\leq n-1}\{\theta_j c_j\}) =$

$$\prod_{j=1}^{n-1}\frac{1}{c_j}\frac{c_1^n}{n} + \prod_{j=2}^{n-1}\frac{1}{c_j}\frac{c_2^{n-1} - c_1^{n-1}}{(n-1)} + \ldots + \prod_{j=k}^{n-1}\frac{1}{c_j}\frac{c_k^{n-k+1} - c_{k-1}^{n-k+1}}{(n-k+1)} + \prod_{j=k+1}^{n-1}\frac{1}{c_j}\frac{1-c_k^{n-k}}{(n-k)} \tag{3}$$

for $1 \leq k \leq n-2$ as specified in (1).

Now we simply have to consider the boundary cases. For $k = 0$, $1 \leq c_1 \leq \ldots \leq c_{n-1}$. Thus,

$$\int_0^1 \prod_{j=1}^{n-1} \Pr(\theta_j \leq \frac{\theta}{c_j})\mathrm{d}\theta = \prod_{j=1}^{n-1}\frac{1}{c_j}\int_0^1 x^{n-1}\mathrm{d}\theta = \prod_{j=1}^{n-1}\frac{1}{c_j}\frac{1}{n}$$

as in (1). For $k = n-1$, $c_1 \leq c_2 \leq \ldots \leq c_{n-1} \leq 1$. The last term in (2) then needs to be reinterpreted as

$$\int_{c_{n-1}}^{1} 1 \ldots 1 \mathrm{d}\theta = (1 - c_{n-1}).$$

□

Proof of Theorem 2. Let expenditures be as specified in the statement of the theorem, and first consider a deviation by one of the inactive contestants. Without loss of generality, suppose contestant 3 deviates to some $x_3 > 0$. There are three cases to consider.

Case i. Suppose that $x_3 \leq x_2^*$. In that case, 3 (weakly) spends least, and with the CSF adapted from (1) their objective function becomes

$$\max_{0 < x_3 \leq x_2^*} \mathbb{E}U_3 = \left(\frac{x_3^2}{3x_1^* x_2^*} \right) V_3 - x_3.$$

Substituting for x_1^* and x_2^* as specified and taking a simple first order condition reveals that $\mathbb{E}U_3$ is initially decreasing and reaches a minimum at $x_3 = \frac{3V_2^3}{8V_1V_3}$. The maximum payoff contestant 3 could achieve by spending x_2^* or less is therefore attained by spending zero if $\frac{3V_2^3}{8V_1V_3} > x_2^*$, meaning a payoff of zero, or by spending x_2^*. But substituting $x_3 = x_2^*$ into the objective function above yields $\mathbb{E}U_3 < 0$ since $V_2 \geq V_3$.

Case ii. Suppose that $x_2^* < x_3 < x_1^*$. In this case, taking the CSF implied by (1), simplifying, and immediately substituting for x_1^* and x_2^* yields the objective function

$$\max_{x_2^* < x_3 < x_1^*} \mathbb{E}U_3 = \left(\frac{x_3}{V_2} - \frac{V_2^3}{12V_1^2 x_3} \right) V_3 - x_3.$$

In this case, the only positive term is $x_3 \frac{V_3}{V_2}$, so since $V_2 \geq V_3$, any $x_3 > 0$ leads to $\mathbb{E}U_3 < 0$.

Case iii. For $x_3 \geq x_1^*$, equation (1) yields the objective function

$$\max_{x_3 \geq x_1^*} \mathbb{E}U_3 = \left(1 - \frac{(x_2^*)^2}{6x_1^* x_3} - \frac{x_1^*}{2x_3} \right) V_3 - x_3.$$

Substituting for x_1^* and x_2^* as specified and taking a simple first order condition shows that $\mathbb{E}U_3$ is maximized when $x_3 = \left(\left(\frac{V_2^3}{12V_1^2} + \frac{V_2}{4}\right) V_3\right)^{1/2}$. If that value is less than x_1^*, then in this case the contestant is best off choosing $x_3 = x_1^*$, yielding $\mathbb{E}U_3 < 0$. Otherwise, substituting the payoff-maximizing value back into the objective function, $\mathbb{E}U_3 < 0$ if and only if

$$V_2 V_3 \left(\frac{V_2^2}{12V_1^2} + \frac{1}{4}\right) < V_2^2 \left(\frac{V_2^2}{6V_1^2} + \frac{1}{2}\right)^2. \tag{4}$$

That must hold because $V_3 \leq V_2$ means that $V_2 V_3 \leq V_2^2$ and the second factor on the left-hand side of (4) is always less than the corresponding factor on the right.

Finally, since all other contestants bid zero, contestants 1 and 2 have no incentive to deviate. □

Proof of Proposition 1.

We first show that in any pure-strategy Nash equilibrium it must be that $x_1^* \geq x_2^*$. (Standard arguments require that x_2^* must lie within the bounds of 0 and V_2, and x_1's limits are similarly established.) Looking at equilibrium payoffs, suppose that instead $x_1^* < x_2^*$, so $\Pr(q_1 > q_2) = \frac{1}{2}\frac{x_1}{x_2}$, and $\Pr(q_2 > q_1) = (1 - \frac{1}{2}\frac{x_1}{x_2})$. The players' objective functions would be,

$$\mathbb{E}U_1 = \frac{1}{2}\frac{x_1}{x_2}V_1 - x_1$$

and

$$\mathbb{E}U_2 = (1 - \frac{1}{2}\frac{x_1}{x_2})V_2 - x_2$$

Taking first order conditions and solving, however, would lead to unique interior solutions of $x_1^* = \frac{V_1^2}{2V_2}$ and $x_2^* = \frac{V_1}{2}$, which contradicts the initial supposition since $V_1 \geq V_2$. Therefore, it must be that $x_1 \geq x_2$.

Knowing that $x_1 \geq x_2$, setting up the players' objective functions as,

$$\mathbb{E}U_1 = (1 - \frac{1}{2}\frac{x_2}{x_1})V_1 - x_1$$

and

$$\mathbb{E}U_2 = \frac{1}{2}\frac{x_2}{x_1}V_2 - x_2,$$

it's then just a matter of taking first order conditions and solving in order to find the pure-strategy equilibrium expenditures and expected payoffs as specified. Given the shape of CSF in this case it is fairly easy to see, but second order conditions also verify that this is indeed the unique pure-strategy equilibrium.

B An Alternative Distribution

While our primary results rest on the assumption that noise is distributed uniformly on $[0, 1]$, in this section we show that similar results hold for the same environment with alternative distributional assumptions. Once again we are interested in the probability that contestant i's final output is greater than that of all others, but we now assume that the noise parameters (the θ_is) are independently distributed on $[0, \infty)$ according to a form of extreme value distribution, a generalized version of the Frechét distribution. Our initial lemma is quite similar to the result of Jia (Theorem 1, 2008), though our proof is different and we offer a slight generalization.

Lemma 1. With noise distributed according to

$$F(x) = \begin{cases} e^{-\theta^{-\alpha}} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \leq 0 \end{cases}$$

α and m,

$$\Pr(q_i > \max_{j \neq n}\{q_j\}) = \frac{x_i^m}{\sum_{j=1}^n x_j^m}.$$

Lemma 1 thus shows that a noisy all-pay auction is equivalent to a ratio-form contest in which the parameter m can be interpreted as the contest's sensitivity to effort–larger values of m mean larger marginal effects of a contestant's efforts on their probability of victory. We can combine this result with a result from Alcalde and Dahm (Proposition 4.1, 2010), who show that any ratio-form contest with $m \geq 2$ possesses an equilibrium in which expected spending and payoffs are the same as in the standard all-pay auction with complete information, to arrive at the following.

Theorem 4. Any all-pay auction with noise distributed according to an inverse exponential distribution with parameters $\alpha > 0$ and $m \geq 2$ and $V_1 \geq V_2 \geq \ldots \geq V_n$ possesses an equilibrium in which expected expenditures are $\mathbb{E}x_1^* = \frac{V_2}{2}$, $\mathbb{E}x_2^* = \frac{V_2^2}{2V_1}$, and $x_j^* = 0$ for all $j > 2$; expected payoffs are $\mathbb{E}U_1^* = V_1 - V_2$, $\mathbb{E}U_2^* = 0$, and $\mathbb{E}U_j^* = 0$ for all $j > 2$; and expected expenditures are $\frac{V_2(V_1+V_2)}{2V_1}$.

In other words, so long as the distribution's m parameter allows the contest to be sensitive enough to effort ($m > 2$), there exists an equilibrium in which all contestants other than those with the highest two valuations abstain from the contest, and those with the two highest valuations behave as they would in an all-pay auction with complete information, with all agents receiving payoffs accordingly. Again, then, even in the presence of noise an all-pay auction's expected results may be unaffected.

C Instructions

The instructions for the treatment with uniform noise on the interval $[0, 1]$ are below. The instructions for the remaining two treatments are available upon request.

Introduction

Welcome to this experiment. The decisions you make during this experiment, as well as the decisions of the participant you will interact with during the experiment, will determine how much money you earn. You will be paid in cash, privately, at the end of our experiment.

It is important that you remain silent and do not look at other people's work. If you have any questions, or need assistance of any kind, please raise your hand and an experimenter will come to you. If you talk, laugh, exclaim out loud, etc., you will be asked to leave and you will not be paid. We expect and appreciate your cooperation.

The following instructions will explain how you can earn money based on the decisions that you, and the other participant you will interact with, make during the experiment. We will go over these instructions with you. After we have read the instructions, there will be time to ask clarifying questions. When we are done going through the instructions, each of you will have to answer a few brief questions to ensure everyone understands.

For today's experiment, you will receive an initial payment of $\$25$, in addition to the $\$7$ show-up fee.

Overview

Throughout today's experiment, numbers on your screen will be rounded to two digits, for simplicity. If we need to compare two numbers, we will compare the unrounded numbers.

In today's experiment you will be randomly matched with another participant. You will interact with this participant in all 40 rounds of the experiment. You will only interact with this other participant through the computer interface, and you will not know who this other participant is.

For each group of two participants, there is a good worth $\$5$ available in each round. Only one of the two participants will obtain this good; it cannot be divided between the two participants in the group.

In each round, both participants in a group will choose an EXPENSE between $\$0$ and $\$5$, inclusive. These EXPENSES will help determine which of the two participants in the group will obtain the available good. Each participant's EXPENSE is unrecoverable. It is subtracted from their payoff for the round, regardless of whether or not they obtain the available good.

No participant will know the EXPENSE chosen by the other participant in their group when they choose their own EXPENSE.

Payoff of a round

In each round, one of the two participants in each group of two will obtain the available good, and one will not.

The payoff for the round for the participant who obtains the good is

$$\$5 - (\text{Their Own EXPENSE}) .$$

The payoff for the round for the participant who does not obtain the good is

$$\$0 - (\text{Their Own EXPENSE}) .$$

Note that payoffs are denominated in dollars.

Random NUMBERS for every participant in every round

In a given round, each participant will have a NUMBER. Each participant's NUMBER is between 0 and 1, inclusive. Each possible NUMBER is equally likely to be chosen. A participant's NUMBER in a given round does not have any effect on his or her NUMBER in any other round, or on the NUMBER of the other participant in their group in any round. That is, NUMBERS are independent across participants and rounds.

Participants will not know their own NUMBER, or the NUMBER of the other participant in their group when they choose their EXPENSE.

Determining which participant in a group obtains the good

To determine which of the two participants in a group will obtain the available good in a given round, the EXPENSE chosen by each participant is multiplied by their own NUMBER. This is called a participant's PRODUCT.

Notice that a participant's PRODUCT is the percentage of his or her EXPENSE corresponding to his or her NUMBER. For example, if a participant's NUMBER is 0.48, then their PRODUCT is 48% of their EXPENSE. Notice that a participant's PRODUCT will be less than their EXPENSE, provided their NUMBER is less than 1.

The participant with the highest PRODUCT in their group will obtain the available good. If the PRODUCTS of the two participants in a group are the same, then the tie is broken randomly, with each participant having an equal chance of obtaining the good.

Remember that the payoff for the round of the participant who obtains the good is

$$\$5 - (\text{Their Own EXPENSE}) .$$

The payoff for the round for the participant who does not obtain the good is

$$\$0 - (\text{Their Own EXPENSE}) .$$

Example 1

Suppose that Participant 1 chooses an EXPENSE of $\$2.92$, and Participant 2 chooses an EXPENSE of $\$3.60$.

Also suppose that Participant 1's NUMBER is 0.75 and Participant 2's NUMBER is 0.30.

Since $2.92 \cdot 0.75 = 2.19$, the PRODUCT of Participant 1 is 2.19. Further, note that 2.19 is 75% of 2.92.

The PRODUCT of Participant 2 is $3.60 \cdot 0.30 = 1.08$. Note that 1.08 is 30% of 3.60.

Since $2.91 > 1.08$, Participant 1 obtains the good. His or her payoff for the round is $\$5 - \$2.92 = \$2.08$. The payoff of Participant 2, who does not obtain the good, is $\$0 - \$3.60 = -\$3.60$.

Example 2

Suppose that Participant 1 chooses an EXPENSE of $\$1.27$, and Participant 2 chooses an EXPENSE of $\$0.75$.

Also suppose that Participant 1's NUMBER is 0.07 and Participant 2's NUMBER is 0.93.

Since $1.27 \cdot 0.07 = 0.0889$, the PRODUCT of Participant 1 is 0.0889. Further, note that 0.0889 is 7% of 1.27.

The PRODUCT of Participant 2 is $0.75 \cdot 0.93 = 0.6975$. Note that 0.6975 is 93% of 0.75.

Since $0.6975 > 0.0889$, Participant 2 obtains the good. His or her payoff for the round is $\$5 - \$0.75 = \$4.25$. The payoff of Participant 1, who does not obtain the good, is $\$0 - \$1.27 = -\$1.27$.

Participating in a round

At the beginning of each round, you will be asked to enter your EXPENSE for the round. Remember that this EXPENSE can be any number between $\$0$ and $\$5$, inclusive.

You will choose your EXPENSE without knowing your NUMBER, the EXPENSE chosen by the other participant in your group or the NUMBER of the other participant in your group.

Results of a round

At the end of each round the results of the round will be displayed on your screen. The results you will see are:

1. Whether or not you obtained the good.
2. Your EXPENSE.
3. Your NUMBER.
4. Your PRODUCT.
5. Your Payoff for the round.
6. The EXPENSE of the other participant in your group.

7. The NUMBER of the other participant in your group.

8. The PRODUCT of the other participant in your group.

9. The Payoff of the other participant in your group.

In addition, the results of all previous rounds will always be displayed on your screen.

Selecting periods for payment

The amount of money you make in today's experiment depends on the decisions you make, as well as the decisions of the other participant in your group. Once all 40 rounds of the experiment have been completed, 10 rounds will be randomly chosen for payment. Each of the 40 rounds are equally likely to be chosen for payment.

Your payment for today's session will be the sum of your payoff in each of the 10 rounds randomly chosen for payment and the initial payment of $\$25$. This is in addition to the $\$7$ show-up fee.

Summary

1. You will be randomly paired with another participant in today's experiment, and you will interact with this participant in all 40 rounds.

2. In each round there is an available good which is worth $\$5$ to both participants in your group.

3. In each round, both participants in your group will choose an EXPENSE, which is a number between $\$0$ and $\$5$, inclusive. You will not know the EXPENSE chosen by the other participant in your group when you make this decision. The EXPENSES chosen by you and the other participant in your group help determine which of you will obtain the available good.

4. If you obtain the available good in a round, then your payoff for the round is $\$5-$(Your Own EXPENSE), and the payoff for the round for the other participant in your group is $\$0-$(Their Own EXPENSE). Similarly, if you do not obtain the available good in a round, then your payoff for the round is $\$0-$(Your Own EXPENSE), and the payoff for the round for the other participant in your group is $\$5-$(Their Own EXPENSE).

5. In every round, each participant in a group will have a NUMBER. Each NUMBER is between 0 and 1 (inclusive), and each possible NUMBER is equally likely to be chosen. The NUMBER of a participant in a given round does not have any effect on the NUMBER chosen in any other round,

or on the NUMBER of any other participant in any round. NUMBERS are independent across participants and rounds.

6. In each round, the EXPENSE chosen by a participant will be multiplied by his or her NUMBER. This is the participant's PRODUCT.
7. In each group and in each round, the participant with the highest PRODUCT will obtain the available good. If there is a tie, it is broken randomly.
8. At the end of the experiment 10 of the 40 rounds will be chosen randomly for payment. Your total payment will be your payoff in these 10 rounds and the initial payment of $\$25$. This is in addition to the $\$7$ show-up fee.

Table 1: Summary statistics of expenditures

Treatment	Observed expenditures	Nash expenditures
Noise on $[0, 1]$	1.645	2.500
	(1.217)	(0.000)
Noise on $[0, 2]$	1.770	2.500
	(0.980)	(0.000)
No noise	1.536	2.500
	(1.671)	(1.443)

Notes: Table contains means with standard deviations in parentheses.

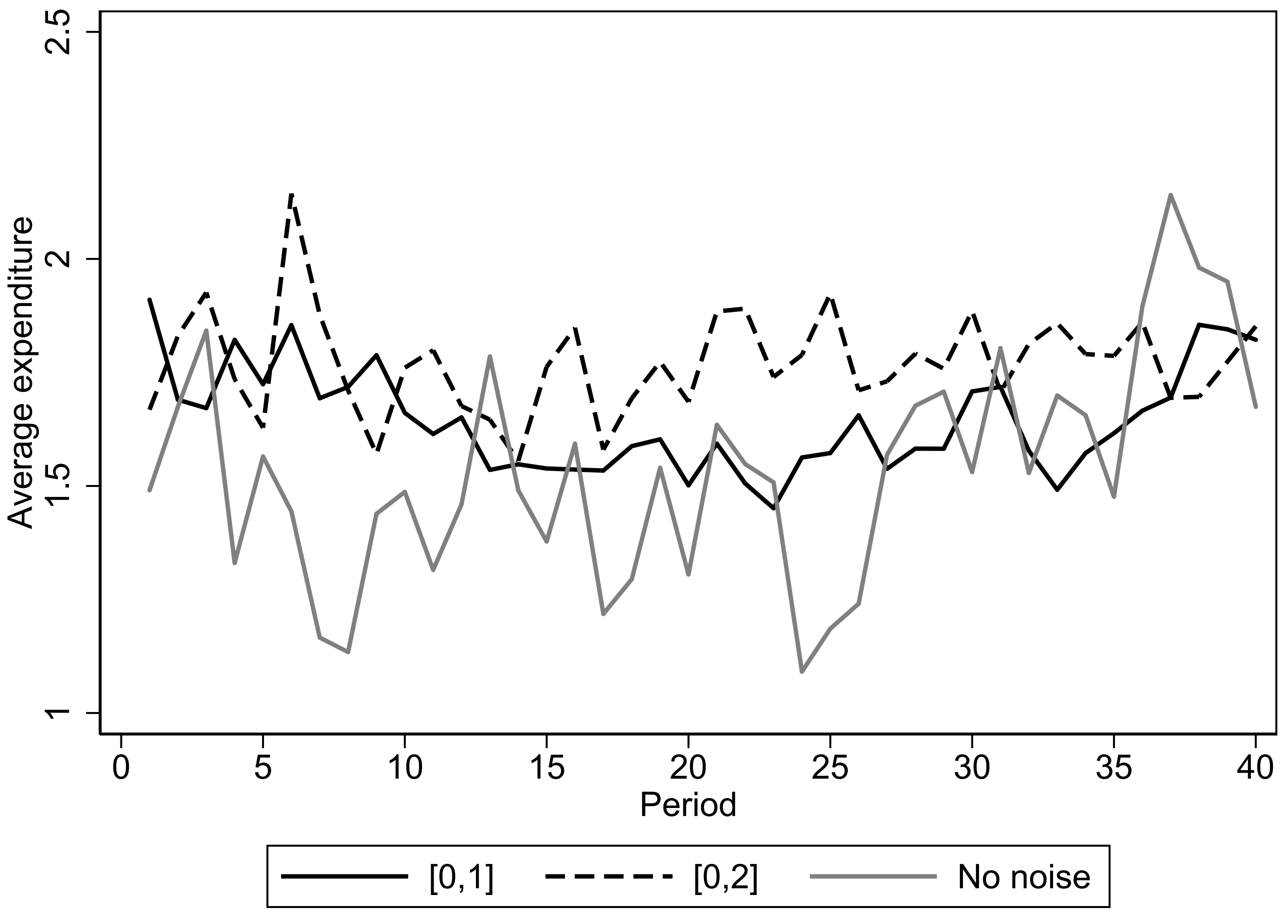


Figure 1: Average expenditures by treatment

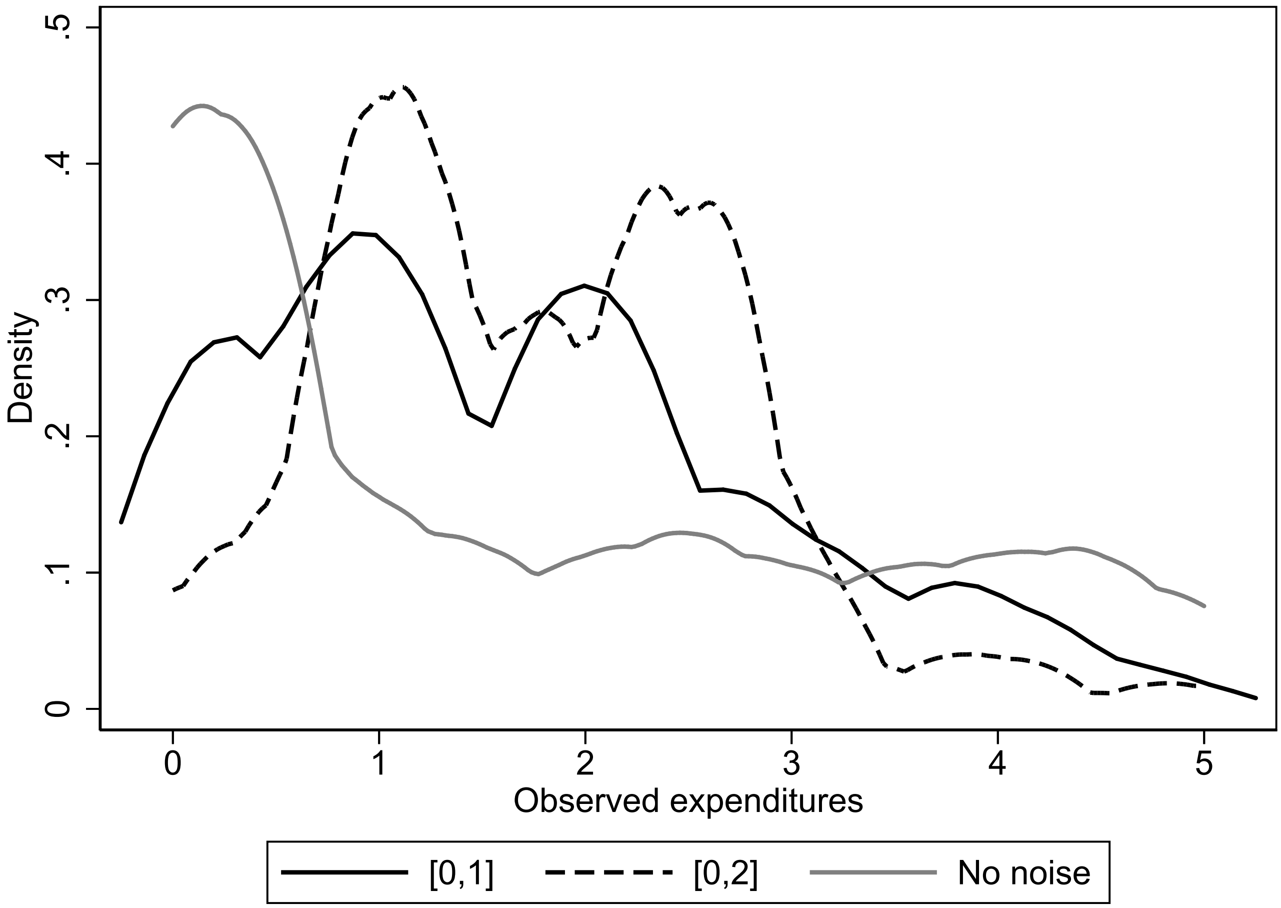
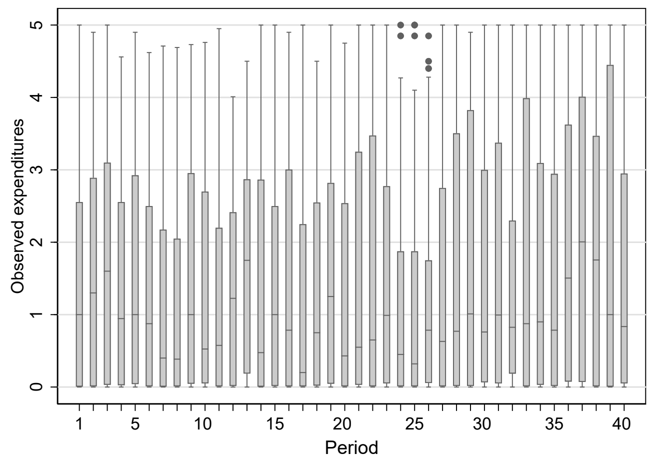
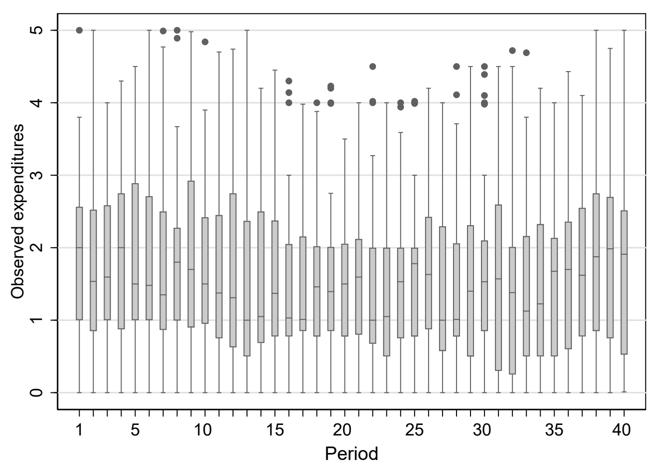


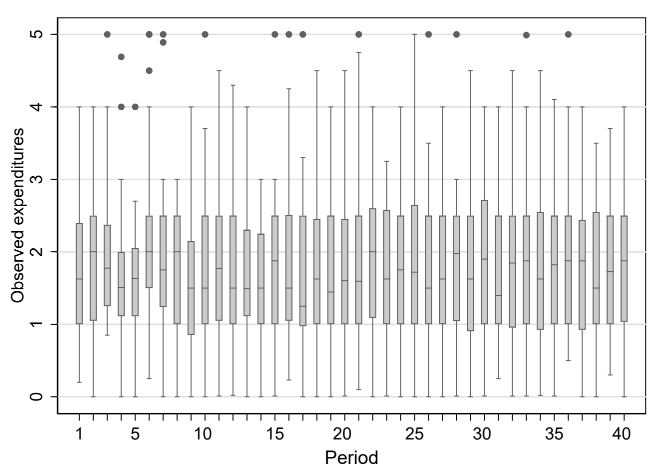
Figure 2: Kernel density plots of observed expenditures by treatment



(a) No noise



(b) Noise on $[\mathbf{0}, \mathbf{1}]$



(c) Noise on $[\mathbf{0}, \mathbf{2}]$

Figure 3: Boxplots of observed expenditures by treatment and period